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Fixed-Pitch Propeller/Piston Aircraft Operations at Partial Throttle

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The linearized propeller polar or bootstrap approach to fixed-pitch propeller aircraft performance is extended to include partial-throttle operations. Power setting parameter Π is defined as the ratio of actual engine torque to maximum possible (full-throttle) torque at the given atmospheric density. To compute quasisteady partial-throttle performance, two additional data items, beyond the nine items of the ordinary bootstrap data plate, are required: 1) brake specific fuel consumption rate as a function of engine power and 2) level cruise speeds with associated engine speeds (from a flight test at any one altitude and weight). The latter results in a propeller chart giving advance ratio J as a function of C_P/J^2, the independent variable in the propeller polar. With that information, level cruise performance tables (similar to GAMA format, but extended with thrust and propulsive efficiency) can be calculated. Installed propeller charts can be constructed and expressions for partial-throttle absolute ceilings derived. Given the airplane's airspeed and desired flight path, e.g., a standard rate turn descending 300 ft/min at 90 KCAS, engine rpm required for that maneuver can then readily be calculated. This partial-throttle facility rounds out the bootstrap approach, allowing full prediction of quasisteady performance of fixed-pitch propeller airplanes.

Nomenclature

A = wing aspect ratio, span^2/area
b = linearized propeller polar intercept
C = altitude engine power dropoff parameter
C_{D0} = parasite drag coefficient
D = drag
d = propeller diameter
E = composite bootstrap parameter
e = Oswald airplane efficiency factor
F = composite bootstrap parameter
G = composite bootstrap parameter
H = composite bootstrap parameter
h = altitude
$\dot{h}$ = rate of climb or descent
J = propeller advance ratio, V/nd
K = composite bootstrap parameter
L = lift
M = engine torque
m = linearized propeller polar slope
n = propeller rps
P = power
Q = composite bootstrap parameter
R = aircraft turn radius; composite bootstrap parameter
S = reference wing area
T = thrust
U = composite bootstrap parameter
V = airspeed
W = gross aircraft weight
γ = flight-path angle
η = propeller efficiency
Π = engine power (throttle) setting
ρ = atmospheric density
σ = relative atmospheric density
Φ = engine torque/power dropoff factor
χ = aircraft bank angle
ω = aircraft turning rate

Subscripts

AC = absolute ceiling
a = available
B = base case, mean sea level standard conditions
bg = best glide
cx = calibrated, best angle of climb
i = induced
max = maximum level flight
md = minimum descent (rate)
min = minimum level flight
p = parasite
r = required
x = best climb angle
xs = excess
y = best climb rate

I. Introduction

STANDARD airplane performance textbooks and monographs seldom discuss partial-throttle operation of propeller-driven aircraft beyond treating best range and best endurance with a simplified theory in which both brake specific fuel consumption (bsfc) rate and propeller propulsive efficiency η are taken as constants. While maximal performance V speeds and maneuvers are important, most of the airplane's life aloft is spent in level cruise. Cruise performance, and other partial-throttle operations, are worthy of greater attention.

Miele[1] mentions a thrust control parameter π for turbojet, turbofan, ramjet, and rocket engines, but none for the propeller airplane. Von Mises[2] treats the partial-throttle subject using an unsupportable approximation in which engine brake moment (torque) is independent of engine speed. If engine speed decreases (e.g., because of increased propeller loading under constant throttle, as during transition from level cruise to climb) engine torque is closely constant, but not when the rpm drop is caused by engine throttling.

When the linearized propeller polar or bootstrap approach to fixed-pitch propeller aircraft performance was first introduced,[3] it was limited to wings-level flight either at full throttle or gliding. The approach was later extended[4] to include full-throttle constant airspeed coordinated turns. This present additional bootstrap extension treats all steady partial-throttle maneuvers (climbs or descents, either wings level or turning) as well as level cruise performance. In our formulation, pulling back to partial throttle is equivalent to replacing the actual engine by a virtual engine of diminished capacity,

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one with reduced mean sea level (MSL) maximum torque. Because throttle is tied to torque, partial throttle means reduced torque. From one of several standard bootstrap formulas, we calculate torque needed for a given airspeed associated either with level cruise or with a known rate or angle of climb or descent. We use a concrete operational definition of power setting:

$$\Pi \equiv M/\,\Phi(\sigma)M_B \tag{1}$$

where M is engine torque at whatever altitude and rpm happen to be the case, $\Phi(\sigma)$ is the altitude power dropoff factor, and M_B is base MSL rated (maximum) torque. The denominator measures maximum possible torque at altitude with full throttle and with fuel–air mixture leaned for maximum power. This definition differs in kind from that of the percent power numbers found in pilots' operating handbook (POH) cruise tables. In them, e.g., 70% brake horsepower (BHP) means 70% of MSL rated full-throttle power at maximum rated rpm. In our definition, $\Pi = 1$ always corresponds to full throttle under ambient conditions.

Under this partial-throttle bootstrap extension, the airplane's steady-state performance is fully described using only the following data: 1) the nine bootstrap data plate parameters for given flaps/gear configuration (to be reviewed); 2) brake specific fuel consumption rate; 3) a graph, table, or curve fit formula specifying advance ratio J as a function of the independent variable in the propeller polar, C_P/J^2 (to be explained); and 4) five operational and pilot control variables: gross weight W, relative air density σ (or density altitude h_ρ), airspeed V, bank angle χ, and power setting Π.

Essentially, the entire formulation of the full-throttle wings-level bootstrap approach is needed for this partial-throttle extension. Therefore, we shall begin by reviewing ordinary bootstrap variables and formulas. Next we lay out theoretical concepts and expressions needed for the partial-throttle extension. Several brief sample calculations will be given of more informative cruise performance table entries and for constructing installed propeller charts. Partial-throttle absolute ceilings are discussed. Unanswered questions and future opportunities close the paper.

II. Assembling the Tools

A. Bootstrap Approach at Full Throttle with Wings Level

To compute quasi-steady-state aircraft performance using the wings-level fixed-pitch normally aspirated version of the bootstrap approach requires the nine aircraft parameters exemplified, for a sample Cessna 172, in Table 1. The proportion of MSL-rated horsepower lost to internal friction, C (always near 0.12), governs full-throttle torque at altitude through the power dropoff factor Φ:

$$M(\Pi = 1, \sigma) = \Phi(\sigma) \times M_B \tag{2}$$

The time-honored form[5] for this dropoff factor is

$$\Phi(\sigma) = (\sigma - C)/(1 - C) \tag{3}$$

Instead of using rated MSL torque directly, one more commonly uses MSL full-throttle power and rated engine rotation speed. The relation is

$$M_B = P_B/2\pi n_B \tag{4}$$

Table 1 Sample bootstrap parameters (Cessna 172)

Bootstrap data plate item	Value	Units	Aircraft subsystem
S	174	ft^2	Airframe
A	7.38	——	Airframe
M_B	311.2	ft lbf	Engine
C	0.12	——	Engine
d	6.25	ft	Propeller
C_{D0}	0.037	——	Airframe
e	0.72	——	Airframe
m	1.70	——	Propeller
b	−0.0564	——	Propeller

Practical work is expedited by using composite bootstrap parameters in forms that either do or do not incorporate explicit weight and altitude dependencies. The four required first-line composites are

$$E = \Phi(\sigma)E_B \quad \text{with} \quad E_B = mP_B/n_Bd \tag{5}$$

$$F = \sigma F_B \quad \text{with} \quad F_B = \rho_0 d^2 b \tag{6}$$

$$G = \sigma G_B \quad \text{with} \quad G_B = \tfrac{1}{2}\rho_0 S C_{D0} \tag{7}$$

$$H = (W/W_B)^2(1/\sigma)H_B \quad \text{with} \quad H_B = 2W_B^2\big/\rho_0 S\pi e A \tag{8}$$

It is often convenient to use second-level composite bootstrap parameters

$$K = \sigma K_B \quad \text{with} \quad K_B = F_B - G_B \tag{9}$$

$$Q = [\Phi(\sigma)/\sigma]Q_B \quad \text{with} \quad Q_B = E_B/K_B \tag{10}$$

$$R = (W/W_B)^2(1/\sigma^2)R_B \quad \text{with} \quad R_B = H_B/K_B \tag{11}$$

$$U = (W/W_B)^2(1/\sigma^2)U_B \quad \text{with} \quad U_B = H_B/G_B \tag{12}$$

Standard weight W_B is usually taken to be the airplane's maximum gross weight. Bootstrap formulas for six V speeds, as true airspeeds in ft/s, are

$$V_{\text{max/min}} = \sqrt{\left(-E \mp \sqrt{E^2 + 4KH}\right)\Big/2K} = \sqrt{-(Q/2) \pm \sqrt{(Q^2/4) + R}} \tag{13}$$

$$V_y = \sqrt{\left(-E - \sqrt{E^2 - 12KH}\right)\Big/6K} = \sqrt{-(Q/6) + \sqrt{(Q^2/36) - R/3}} \tag{14}$$

$$V_x = [(-H)/K]^{\frac{1}{4}} = (-R)^{\frac{1}{4}} \tag{15}$$

$$V_{\text{bg}} = (H/G)^{\frac{1}{4}} = U^{\frac{1}{4}} \tag{16}$$

$$V_{\text{md}} = (H/3G)^{\frac{1}{4}} = (U/3)^{\frac{1}{4}} \doteq 0.7598 V_{\text{bg}} \tag{17}$$

Those six V speed results (see Ref. 3 for details) come directly from the following:

$$P_a \equiv TV = EV + FV^3 \tag{18}$$

$$P_r \equiv DV = GV^3 + H/V \tag{19}$$

$$P_{xs} \equiv P_a - P_r = EV + KV^3 - H/V \tag{20}$$

$$T(V) = E + FV^2 \tag{21}$$

$$D(V) = D_p(V) + D_i(V) = GV^2 + (H/V^2) \tag{22}$$

$$T_{xs} \equiv T - D = E + KV^2 - (H/V^2) \tag{23}$$

In addition, it is important to have formulas for

$$ROC(V) = \dot{h}(V) = \frac{P_{xs}(V)}{W} = \frac{EV + KV^3 - H/V}{W} \tag{24}$$

$$\gamma(V) = \sin^{-1}\frac{T_{xs}(V)}{W} = \sin^{-1}\left(\frac{E + KV^2 - H/V^2}{W}\right) \tag{25}$$

The wings-level composite parameters are evaluated, for four different situations that will be used in examples throughout this paper (Table 2).

Next we turn to computation of the airplane's absolute ceiling. The mathematical prescription for absolute ceiling is that the $P_a(V)$

Table 2 Sample composite bootstrap parameters, four situations

Variable or Composite	Case 1	2	3	4
h_ρ	MSL	3,000 ft	6,000 ft	12,000 ft
W	2,400 lbf $= W_B$	2,300 lbf	2,400 lbf $= W_B$	2,400 lbf $= W_B$
σ	1.0000	0.9151	0.8359	0.6932
$\Phi(\sigma)$	1.0000	0.9035	0.8135	0.6513
E	531.911 $= E_B$	480.60	432.70	346.45
F	−0.0052368 $= F_B$	−0.0047923	−0.0043772	−0.0036300
G	0.0076516 $= G_B$	0.0070021	0.0063956	0.0053039
H	1,668,535 $= H_B$	1,674,526	1,996,192	2,407,100
K	−0.0128884 $= K_B$	−0.0117944	−0.0107729	−0.0089339
Q	−41,270.6 $= Q_B$	−40,748.5	−40,165.4	−38,779.5
R	−129,460,301 $= R_B$	−141,976,614	−185,297,929	−269,435,170
U	218,064,595 $= U_B$	239,147,235	312,118,214	453,840,065

and $P_r(V)$ curves, given by Eqs. (18) and (19), touch at a single point. And because those curves are smooth (have continuous derivatives), their slopes are the same at that point of osculation. This dual condition gives the wings-level full-throttle results[6]

$$\Phi_{\mathrm{AC}}(W, \Pi = 1, \chi = 0) = (2W/\, W_B E_B)\sqrt{-H_B(0)K_B} \quad (26)$$

$$V_{\mathrm{AC}}(W, \Pi = 1, \chi = 0) = (W/\, W_B)\sqrt{[2H_B(0)/\,\sigma_{\mathrm{AC}}\Phi_{\mathrm{AC}}E_B]} \quad (27)$$

where E_B means $E(\Pi = 1, \sigma = 1)$ and σ_{AC} is given by inverting the usual Gagg–Farrar[5] altitude dropoff relation [Eq. (3)]. The airspeed given by Eq. (27) is V_x (or indeed V_y or $V_{\max}$ or $V_{\min}$, because all have coalesced, at absolute ceiling, to a single value). Later, we shall extend and simplify Eqs. (26) and (27) to more general banked ($\chi > 0$) or partial-throttle ($\Pi < 1$) cases.

B. Bootstrap Approach Extended to Full-Throttle Banked (Turning) Flight

To extend the wings-level bootstrap approach to include constant-airspeed coordinated turns, all that is necessary is the substitution

$$H \equiv H(0) \rightarrow H(\chi) \equiv H(0)/\cos^2\chi \quad (28)$$

The parenthesized zero, as in Eqs. (26–28), denotes a value for unbanked, wings-level flight. For most intents and purposes, banking to angle χ is tantamount to increasing gross weight from W to $W/\cos\chi$.

III. Extension of the Bootstrap Approach to Partial-Throttle Operations

Unlike extension of the full-throttle wings-level theory to include maneuvering (which required only a slight emendation of the induced drag term), extension to partial-throttle operations requires additional data on both engine and propeller. To calculate fuel usage, we must know bsfc; that much is easy. For our sample Cessna 172, with its Lycoming O-320-D2J engine, as for almost all general aviation aircraft, the appropriate engine manual is readily available and includes the necessary graph, from which we obtain[7]

$$c = \begin{cases} 0.45, & \mathrm{hp} \le 122 \\ 0.51, & \mathrm{hp} > 122 \end{cases} \quad (29)$$

The units of c are lbm/h/hp. Aviation gasoline density is approximately 6 lbm/gal (U.S.).

Part of the necessary propeller knowledge might be gleaned from previously ignored rpm data taken during bootstrap data plate test flights. But a new partial-throttle level flight test, at one known weight and density altitude, will also be required. The goal (not motivated at this point, but discussed later in this section) is to obtain a graph, table, or curve fit formula for advance ratio J as a function of $C_P/\,J^2$. The train of thought and calculation runs as follows. Because the propeller power coefficient is defined as

$$C_P \equiv \frac{P}{\rho n^3 d^5} \quad (30)$$

and power is connected to torque by

$$P = 2\pi n M \quad (31)$$

we see that

$$\frac{C_P}{J^2} = \frac{2\pi M}{\rho d^3 V^2} \quad (32)$$

Therefore if (having chosen airspeed V) we can find torque M, we could enter our as yet unavailable graph to get J. Then it would simply be a matter of using

$$n = V/\,dJ \quad (33)$$

to find engine speed n (or engine rpm as $N = 60n$). Then this rpm data would let us calculate engine power, from torque, and then fuel flow. In fact, rpm information allows us to do much more.

Of the bootstrap first-line composite parameters (E, F, G, and H), torque only appears in the definition of E. Using Eqs. (4) and (5)

$$M(\Pi = 1, \sigma) = \Phi(\sigma)M_B = \frac{\mathrm{d}E(\Pi = 1, \sigma)}{2\pi m} \quad (34)$$

Finding composite E will thereby immediately get torque. We have several equations, most notably Eq. (13) for either $V_{\max}$ or $V_{\min}$ (in the $\Pi < 1$ case, V_{cr}), Eq. (24) for rate of climb, and Eq. (25) for angle of climb, which allow an easy solution for composite parameter E. But there is an apparent problem: E and M have thus far been associated only with full-throttle operations. Values of M smaller than base-rated torque M_B have certainly been found at full throttle, but those have been smaller only because of altitude dropoff factor $\Phi(\sigma)$. How then is the needed extension possible?

A basic notion and approximation of this paper is that an engine at partial throttle can be thought of as a smaller engine at full throttle. If someone took a hacksaw and cut an inch off your throttle control plunger and screwed the knob back on over the cut, you would suddenly have an airplane of diminished performance. But a neophyte pilot would have no easy way of distinguishing between this sabotaged partial-throttle situation and a virtual engine characterized by reduced base MSL full-throttle torque M_B'. The reader should keep in mind that we are implying the possibility of many different virtual engines, each with its own derated MSL torque value M_B'. Which particular virtual engine we use depends on assumed airspeed and flight path. We also assume that any such virtual engine has the same power dropoff parameter C as does the real one. That assumption is supported by the fact that $C \doteq 0.12$ for aircraft engines over a considerable range of total displacements.

At a given airspeed, and given that one is either in level cruise or climbing or descending at a given rate or with a given flight-path angle, Eqs. (13), (24), and (25) cover the partial-throttle operational possibilities. Each of those equations can be solved for composite E, and, therefore, for torque M. In the level cruise case, $V = V_{\mathrm{cr}}$, Eq. (13) gives

$$M(\Pi, \sigma) = \Phi(\sigma)M_B'(\Pi, \sigma = 1) = \frac{R - V_{\mathrm{cr}}^4}{V_{\mathrm{cr}}^2} \times \frac{\sigma K_B d}{2\pi m} \quad (35)$$

Equations (32) and (35), together with Eq. (15) tying R to V_x (which V speed incidentally is independent of throttle setting), allow us to eliminate torque M to obtain

$$\frac{C_P}{J^2} = \left[\left(\frac{V_x}{V_{\mathrm{cr}}}\right)^4 + 1\right] \times \frac{(-K_B)}{\rho_0 d^2 m} \tag{36}$$

For given level flight speed V_{cr}, we can find C_P/J^2. Further progress requires a brief one-time level flight test, recording pairs (KCAS, rpm), to give associated values of the advance ratio J.

We return to the question of which variation of Eq. (13) to use, the form for maximum level flight speed, or for minimum. Solution Q, and then M, is the same for either. Being able to ignore that distinction implies a relationship between $V_{\max}$, $V_{\min}$, and R and it is simple:

$$V_x = \sqrt{V_{\max} V_{\min}} \tag{37}$$

where V_x is the geometric mean between bootstrap minimum and maximum speeds for level flight.

In the climbing or descending cases, Eqs. (24) and (25) give

$$M(\Pi, \sigma) = \Phi(\sigma) M_B'(\Pi, \sigma = 1)$$

$$= \begin{cases} \left[\dfrac{W\dot{h}}{V} + \sigma K_B\left(\dfrac{R - V^4}{V^2}\right)\right]\dfrac{d}{2\pi m} \\ \left[W \sin\gamma + \sigma K_B\left(\dfrac{R - V^4}{V^2}\right)\right]\dfrac{d}{2\pi m} \end{cases} \tag{38}$$

A possible objection to our virtual derated engine procedure may be that linearized propeller polar parameters m and b were calculated using data from full-throttle flight tests. How can one be sure the same values of those parameters will be obtained in the partial-throttle case? A more involved answer to this objection comes from reviewing formulas from which m and b were originally calculated [Ref. 3, Eq. (18) for b, and either Eq. (19) or (20) for m]. One sees that the relation for b does not involve engine torque or any variable that depends on engine torque. And while either relation for m does indeed involve base engine torque M_B, either could just as well be looked upon as a relation for the product mM_B. In other words, if one had originally inadvertently underestimated M_B, one would have calculated, as a consequence, a larger and precisely compensating value for m. A shorter, and likely more satisfying, answer to the objection comes from remembering that the propeller polar is a relation having only to do with the propeller. One of the most useful features of the bootstrap approach is its separation of the airplane into three essentially independent parts: the airframe, engine, and propeller. Utility of dimensionless coefficients and other quantities in the propeller polar hinges on the fact that their values do not depend on the particular engine employed to determine them.

At this point, for definiteness and for use in sample calculations, we describe a prescription from which one may determine $J(C_P/J^2)$. Take the airplane to a given density altitude at a known gross weight (these may be calculated after the flight) and a given flaps/gear configuration. Fly level and stabilized at various (indicated) airspeeds and record the associated engine speeds. The AGARD manual[8] suggests taking 3–10 min to let the aircraft stabilize at each level airspeed and suggests moving down from the highest speed instead of up from the lowest. Using the airplane's airspeed-indicator calibration curve, correct indicated airspeeds to calibrated airspeeds. As an example, consider our sample Cessna 172 with flaps up at density altitude 3000 ft, gross weight 2300 lb (Table 2, case 2). Data as in Table 3 (fabricated from propeller charts for the McCauley 7557 propeller on this airplane and an assumed installed slowdown factor of 8.5% as suggested by Norris and Bauer[9]) are collected (first and third columns) and lightly massaged (second and fourth columns).

With this raw and immediately derived performance flight test data in hand, Eq. (36), together with standard bootstrap information on the pertinent value of V_x (bootstrap composite parameter

Table 3 Supplementary level cruise flight test data

KCAS	KTAS	N	n
50	52.27	2021	33.68
55	57.49	1959	32.65
60	62.72	1946	32.43
65	67.95	1967	32.78
70	73.17	2013	33.54
75	78.40	2076	34.60
80	83.63	2153	35.88
85	88.85	2239	37.32
90	94.08	2332	38.87
95	99.31	2431	40.52
100	104.54	2534	42.24
105	109.76	2640	44.01

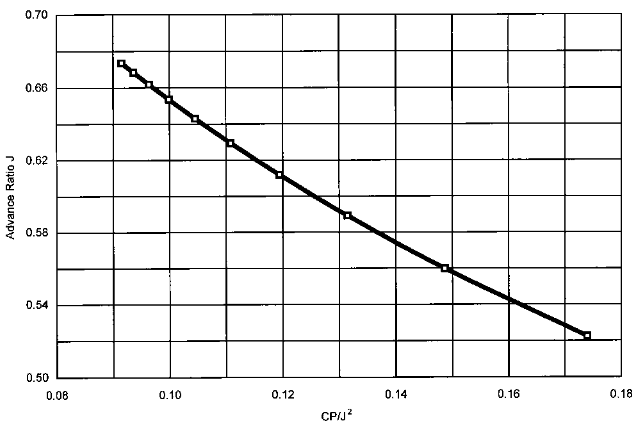

Fig. 1 Advance ratio J as a function of C_P/J^2 for the McCauley 7557 propeller.

$R = -141{,}976{,}614$ corresponds to $V_x = 109.2$ ft/s = 64.7 KTAS) provides corresponding values of C_P/J^2. A graph such as Fig. 1 results. To be able to calculate with the graph, it is convenient (although devoid of any deeper meaning) to fit a curve to it. In our sample case, one such fit is given by

$$J = \left(6.9145 - 5.9501e^{-C_P/J^2}\right)^{-1} \tag{39}$$

In fact, the range of J values given by the data in Table 3 is somewhat too small to cover some sample calculations shown later in this paper. That is where, in practice, reconsideration of early full-throttle climbing data, or even an additional flight test employing partial-throttle descents, timed over some vertical interval, might come in handy. Instead of belaboring experimental exigencies, we will simply use the curve fit of Eq. (39) as though it subsumed all of our requirements. This somewhat unusual but monotonous propeller chart [Fig. 1 or Eq. (39)] is the major result gained by the supplementary level cruise flight test. But additional valuable information, propeller power and thrust coefficient graphs, and a propulsive efficiency graph, can also be gleaned from it. We next demonstrate its use, within our theory, with sample calculations.

IV. Cruise Performance Tables Using the Partial-Throttle Extension

A. GAMA-Format Cruise Performance Table

Such a table is for a given type airplane in a given flaps/gear configuration, wings level, at a given gross weight. The table is entered first with pressure altitude and outside air temperature, then with any of several values of engine speed N in rpm. Dependent data consist of percent rated MSL brake power, true airspeed V in KTAS, and fuel flow rate in gallons per hour (gph). Table 4 shows POH cruise performance entries, for one density altitude, for the Cessna 172 airplane without speed fairings. Because fuel flow comes directly from brake power, it is convenient to cut entries for each density altitude back to triples (N, V, P). Although any one of those three might be considered a second independent variable, that role usually falls to N. It will turn out, in our alternative bootstrap cruise table, that N is not a good choice for independent variable.

B. Calculation of Additional Bootstrap Cruise Performance Information

Brief bootstrap calculations, and the use of density altitudes rather than the usual pressure altitude/outside air temperature combination, make it easy to construct a cruise performance table with considerably more information though with fewer columns than the typical POH table. Table 5 is a section for our sample Cessna.

In theoretical developments we use torque M, rather than power P, because torque is most closely tied—through manifold absolute pressure (MAP), and brake mean effective pressure (bmep)—to throttle position. And partial-throttle operation is our subject. Therefore, we will consider the slightly transformed cruise performance table that has, for each of several density altitudes, entries (V, M, N). To calculate two of those from the third, two functions of single variables are required. The first, of form $M(V)$, is Eq. (35). The second, of form $N(V)$ [only for our particular curve fit, Eq. (39)] is

$$N = (60V/d)\left(6.9145 - 5.9501e^{-C_P/J^2}\right) \tag{40}$$

where C_P/J^2 comes from Eqs. (36) or (32).

Here are details for our sample Cessna 172, flaps up, 2400 lb, at 6000 ft, for cruise air speed $V = 95$ KTAS $= 160.3$ ft/s. From Table 2 (case 3), the altitude factors are $\sigma = 0.8359$ and $\Phi = 0.8135$, $V_x = (-R)^{1/4} = 185{,}297{,}929^{1/4} = 116.7$ ft/s. From Eq. (35), torque $M = 207.5$ ft lbf. From Eq. (32), $C_P/J^2 = 0.1046$. From Eq. (40), $N = 2393$ rpm. With circular speed we can then calculate engine power $P = 2\pi nM = 52{,}007$ ft lbf/s $= 94.56$ hp. Because the rated MSL power is 160 hp, the engine is developing 59.1% of full power. At this power level, from Eq. (29), bsfc $= 0.45$ lbm/h/hp; fuel-consumption rate is 7.09 gph. From Table 4 we see that the POH Cessna 172, at this same rpm, is almost 10 KTAS faster. Our sample airplane is neither as clean nor as efficient as the Cessna POH prototype.

The specimen cruise performance table (Table 5) illustrates several points.

1) At low speeds, some rpm values correspond to either of two airspeeds. Therefore, N is not an appropriate independent variable for a widespan cruise performance table. This defect does not show in current POH cruise tables because those cut off at relatively moderate airspeeds. (Considering that the POH Cessna is about 10 KTAS faster than our sample airplane, the POH cruise table only goes down to settings for approximately our 78 KTAS.) Airspeed is the appropriate independent variable because that choice makes the other performance items single valued. We chose calibrated (equivalent) airspeed as closest to the operational indicated airspeed.

2) KTAS is included, as in the POH cruise tables, because of its usefulness in navigation.

3) Power setting Π is a new feature. It is calculated either as M'_B/M_B, which would be $(207.5/0.8135)/311.2 = 82.0\%$ for the recent 95 KTAS example, or as $M/\Phi M_B = 207.5/(0.8135 \times 311.2) = 82.0\%$.

4) Thrust can be calculated, if desired, from Eq. (21). Because the craft is at partial-throttle, $E(\Pi, \sigma = 1) = \Pi E(\Pi = 1, \sigma = 1)$, is the correct base composite factor to use. Because this is level flight, using Eq. (22) for drag would avoid even considering throttle setting.

5) Propeller efficiency η is another new feature. In the next section we show how to plot installed propeller propulsive efficiency $\eta(J)$ from ordinary and extended bootstrap flight tests. Because our extended flight-test data were fabricated, efficiency calculated from thrust, airspeed, and power gives results a few percent higher than that in Table 5. Accurate experimental data would give more consistent calculations.

6) It seems imperative that a cruise performance table contain V_{br} (speed for best range in calm air) and V_{be} (speed for best endurance). V_{br} is found, by calculation and inspection, as the speed for which V/cP is maximum; V_{be} is the speed for which $1/cP$ is maximum. In the usual simplified theory, in which both propeller efficiency η and specific fuel consumption c are taken to be constant, V_{br} (maximum C_L/C_D) is speed for best glide V_{bg} [Eq. (16)], and V_{be} (maximum $C_L^{3/2}/C_D$) is the speed for a minimum descent rate V_{md} [Eq. (17)]. Our calculations improve realism in that η varies with airspeed, and closely following the engine manual for the Cessna 172's Lycoming O-320-D2J engine, c is taken [Eq. (29)] to be only piecewise constant. For our sample Cessna at 6000 ft, 2400 lb, $V_{bg} = 72.0$ KCAS, only 1 kn off our $V_{br} = 73$ KCAS, but $V_{md} = 54.7$ KCAS, more than 7 KCAS below our best endurance speed $V_{be} = 62$ KCAS. That difference is a result of the moderately sharp decrease in propeller efficiency at lower values of J. Pilots should be able to base crucial range or endurance decisions on fuller data, as in Table 5.

V. Installed Propeller Thrust and Power Coefficients and Efficiency Graphs

Propeller blade elements, under steady conditions, move in two mutually perpendicular directions, tangentially and longitudinally, requiring two independent scalar propeller relations. The linearized propeller polar

$$C_T/J^2 \doteq m\left(C_P/J^2\right) + b \tag{41}$$

is one of these, but, because it is only one, it can never do the full job. That is why the supplementary level cruise flight test (or some other test recording engine speed) is necessary. Considering what we gain from it, performing that brief test is a small price to pay.

With additional flight tests performed and the data processed, one makes graphs of propeller coefficients $C_P(J)$ and $C_T(J)$ and of propulsive efficiency $\eta(J)$ for those values of advance ratio J

Table 4 Cessna 172 POH cruise performance at 6000 ft

rpm	% bhp	KTAS	gph
2600	77	117	8.6
2500	69	111	7.8
2400	63	105	7.0
2300	57	99	6.4
2200	52	93	5.9
2100	47	86	5.5

Table 5 Bootstrap expanded cruise performance table for Cessna 172, 2400 lb, 6000 ft, flaps up

KCAS	KTAS	N, rpm	P, %	gph	Π, %	T or D, lbf	η
100	109.4	2662	78.9	10.7	98.4	284	0.735
95	103.9	2556	70.5	8.5	91.6	298	0.739
90	98.4	2454	63.2	7.6	85.4	312	0.744
85	93.0	2359	56.9	6.8	80.0	325	0.748
80	87.5	2272	51.7	6.2	75.5	337	0.751
75	82.0	2196	47.5	5.7	71.8	349	0.751
73 V_{br}	79.8	2169	46.2	5.5	70.6	353	0.750
70	76.6	2134	44.5	5.3	69.2	360	0.747
65	71.1	2093	42.8	5.1	67.9	370	0.735
62 V_{be}	67.8	2081	42.5	5.1	67.8	375	0.722
60	65.6	2079	42.7	5.1	68.1	379	0.711
55	60.2	2104	44.6	5.4	70.4	388	0.671

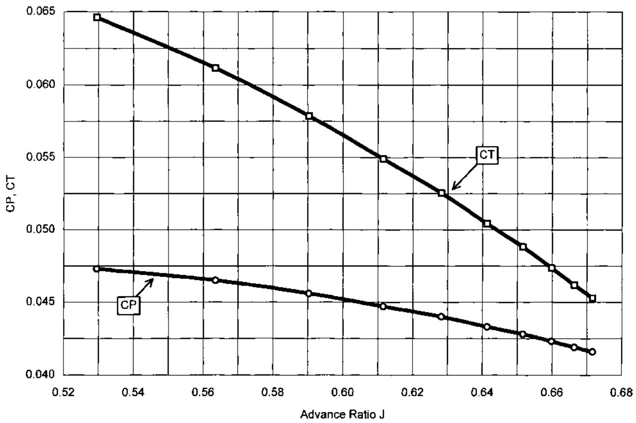

Fig. 2 Thrust and power coefficient functions for the McCauley 7557 propeller.

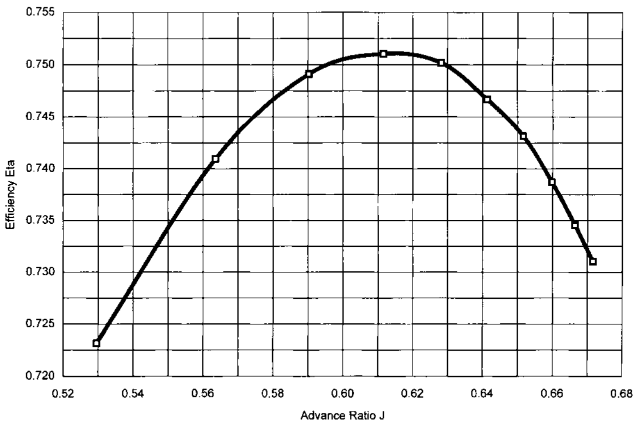

Fig. 3 Inferred propulsive efficiency for the McCauley 7557 propeller installed on a Cessna 172.

encountered in the tests. The power coefficient, because we have pairs $(C_P/J^2, J)$, is immediate by multiplying abscissas by J^2. Next, we can use our linearized propeller polar [Eq. (41)], to calculate matching values of C_T/J^2. From those one finds $C_T(J)$. Then we get matching values of propulsive efficiency from

$$\eta(J) = J \times \frac{C_T/J^2}{C_P/J^2} \tag{42}$$

Figure 2 shows the thrust and power coefficient functions as determined by our theory, and Fig. 3 gives the corresponding propulsive efficiency function. The graphs are limited here to the experimentally accessible range of level cruise J values.

It might be said that these tortuous peregrinations are easily circumvented by simply using the propeller manufacturer's charts for the same quantities. But that would ignore the institutional fact that propeller charts are generally treated as proprietary information and are seldom made available to ordinary pilots or fleet operators. This would also ignore the technical fact that even if charts were available, they would only be for the uninstalled propeller. Once installed, the fuselage modifies airflow through the propeller, and, hence, the propeller's action and efficiency.

VI. Calculating Engine RPM Required to Follow a Given Flight Path

A. Partial-Throttle Operation at Given Speed, Turn Rate, and Climb or Descent Rate

The best way to demonstrate the calculation wrinkles demanded by climbing or descending turns (including wings-level climbs and descents as a special case) is to go through an example. In this subsection we consider our sample Cessna 172 at 6000 ft, 2400 lb, flaps up, making a standard rate (3 deg/s) turning 300 fpm descent at 90 KCAS. Here there are steps and formulas required to compute bank angle χ, engine speed N, and further performance items of interest.

1) Use the standard turn relation

$$\chi = \tan^{-1}(\omega V/g) \tag{43}$$

Three deg/s is 0.05236 rad/s and $V = 90$ KCAS $= 98.44$ KTAS $=$ 166.15 ft/s. Hence, $\chi = 15.13$ deg. This will modify bootstrap composite H for additional induced drag.

2) A useful variation of the top portion of Eq. (38), directly using composite parameter values as in Table 2, is

$$M(V, \dot{h}, \chi, W, \sigma) = \frac{d}{2\pi n m}\left[\frac{\dot{h}W}{V} - KV^2 + \frac{H(0)}{\cos^2\chi V^2}\right] \tag{44}$$

Substituting the proper bootstrap data plate and composite items (case 3 of Table 2) and bank angle gives $M = 177.15$ ft lbf. (Recalling altitude power dropoff factor Φ, at 6000 ft, is 0.8135, the MSL base torque is 311.2 ft lbf, and the power setting is $\Pi = M/\Phi M_B = 70\%$.)

3) Use Eq. (32) to find that $C_P/J^2 = 0.0831$. Then, from Eq. (40), $N = 2295$ rpm. The desired descending turn, maintaining 90 KCAS, results from banking 15 deg and throttling back to ~2300 rpm.

4) From the linearized propeller polar [Eq. (41)], $C_T/J^2 = 0.0849$, and so, propeller efficiency, during the early stages of this partial-throttle descent, using Eq. (42) with $J = 0.6950$, is $\eta = 0.6950 \times 0.0849/0.0831 = 71.0\%$.

B. Partial-Throttle Descent at Given Speed Along Given Approach (Glide) Path

Again, consider the sample Cessna 172 at 2400 lb at 6000 ft with flaps up. What rpm is (initially) required for the air-plane to descend along a 3 deg glide slope (in calm air) at 90 KCAS?

The glide slope is -0.0524 rad, with sine -0.0523. The formula similar to Eq. (44), but featuring a flight-path angle is

$$M(V, \gamma, \chi, W, \sigma) = \frac{d}{2\pi m}\left[W\sin\gamma - KV^2 + \frac{H(0)}{\cos^2\chi V^2}\right] \tag{45}$$

In our example, χ is zero. $M = 142.88$ ft lbf [hence $\Pi = 142.88/(0.8135 \times 311.2) = 56.4\%$]. $C_P/J^2 = 0.0670$, by Eq. (32), and $N = 2153$ rpm by Eq. (40). To descend along the 3-deg glide slope, at 90 KCAS, the pilot should throttle back to ~2150 rpm.

A continuing question is, assuming that the indicated airspeed and physical throttle position are maintained down to 2000 ft, and that the latter implies a constancy of Π, what will the rpm be at this lower altitude? At 2000 ft, $\sigma = 0.9428$, and altitude dropoff factor $\Phi = 0.9350$. True airspeed has decreased to 92.7 KTAS $=$ 156.4 ft/s. Torque M will be increased by the ratio of the dropoff factors, $0.9350/0.8135 = 1.15$, to 164.31 ft lbf. C_P/J^2 is also increased by that factor because airspeed in the denominator of Eq. (32) is effectively calibrated (equivalent). Then $C_P/J^2 = 0.0771$. By Eq. (40), $N = 2110$ rpm. Engine speed has therefore dropped by only 10 rpm per thousand feet; this is difficult to see in even very calm air.

VII. Partial-Throttle Absolute Ceilings

Because it is well known that an airplane cannot possibly achieve its absolute ceiling in finite time with finite fuel, unless deposited there by some higher-flying entity (see Ref. 2, p. 461), the calculation of absolute ceilings and the speeds there appears pointless, at most being a comparative benchmark. But, in fact, getting to absolute ceiling is impossible only when one is restricted to wings-level full-throttle flight. Banking or throttling back does not give the airplane an upward boost; those control manipulations work the other way around, dragging the absolute ceiling down toward current altitude. So absolute ceiling, in relative terms, is attainable by flying high, full throttle, with wings level, and then banking or throttling. In so reducing the ceiling, some interesting facts emerge. Moving from the full-throttle wings level ceiling relations [Eqs. (26) and

(27)], to the possibly banked, possibly partial-throttle case is easy. Using the general prescription for banking [Eq. (28)], and the similar prescription for partial throttle

$$E_B \to E'_B(\Pi, \sigma = 1) = \Pi E_B \tag{46}$$

along with the composite formula for V_x, Eq. (15), one obtains

$$\Phi_{AC}(W, \Pi, \chi) = \frac{1}{\Pi \cos\chi} \Phi_{AC}(W, \Pi = 1, \chi = 0) \tag{47}$$

$$V_{CAC}(W, \Pi, \chi) = \sqrt{\frac{W}{W_B \cos\chi}} V_{cxB} \tag{48}$$

If, for instance, one goes from full to three-fourths throttle ($\Pi = 0.75$), ceiling dropoff factor Φ increases by a third, and ceiling density altitude is considerably diminished, for example, from 16,000 ft ($\Phi = 0.5556$) to 8582 ft ($\Phi = 0.5556 \times \frac{4}{3} = 0.7409$). Similarly, if one banks 41.4 deg the result is the same. If the airplane is currently above the new banked or throttled ceiling, the airplane descends to it.

We have chosen to express the airspeed required at the ceiling [Eq. (48)] in calibrated terms to simplify matters. V_{cxB} is the calibrated speed for the best angle of climb under base conditions. For a given airplane in a given configuration, it is just a number, independent of altitude. The interesting aspect of Eq. (48) is that it shows calibrated airspeed at absolute ceiling, whereas it does depend on weight and bank angle, and is independent of ceiling altitude and of power setting Π. Throttling back lowers the ceiling, but the calibrated airspeed needed to maintain level flight at that lower ceiling is precisely the same as before.

For a fuller numerical example, consider the airplane described in Table 2, case 4. Suppose that, wings level at 12,000 ft, the pilot throttles back to $\Pi = 0.75$. What will the absolute ceiling, and the calibrated airspeed to maintain it, then become? From Eqs. (47) and (26), using composite parameter values from Table 2, $\Phi_{AC}(W, \Pi, \chi) = \Phi_{AC}(2400, 1, 0) = 0.5514$. Because the pilot then goes to three-quarters throttle, the new $\Phi_{AC}(W, \Pi, \chi) = \Phi_{AC}(2400, 0.75, 0) = 0.7352$. Inverting Eq. (3), one finds that this corresponds to $\sigma_{AC}(W, \Pi, \chi) = \sigma_{AC}(2400, 0.75, 0) = 0.7670$ which, by

$$h_\rho = 145{,}457(1 - \sigma^{0.23494}) \tag{49}$$

gives $h_{\rho AC} = 8790$ ft. Absolute ceiling airspeeds are always V_x and, because this speed for best angle of climb does not vary (in calibrated terms), when the wings are kept level at a constant weight, it is convenient to compute at the original 12,000 ft altitude: $V_x = (-R)^{1/4} = 269{,}435{,}170^{1/4} = 128.12$ ft/s $= 75.91$ KTAS. Using the value for relative air density at that altitude, $\sigma = 0.6932$, $V_{cx} = 63.2$ KCAS. Having pulled back to $\Pi = 0.75$, the airplane descends to 8790 ft when maintained at optimum airspeed at 63.2 KCAS.

The importance of V_x, and particularly of V_{cx}, runs like a red thread through the bootstrap approach to propeller aircraft performance. Numerous investigators' schemes for simplified performance calculations[10,11] made use of other dimensionless speed ratios. More often than not, base speed was chosen as maximum level flight speed (either at sea level or at altitude) or as level flight speed for maximum propeller efficiency. Very often those were taken to be the same. An exception is the treatment of Perkins and Hage,[12] which uses V_{bg} as the base speed. V_{bg} is close to V_x in concept, and, in fact, V_{bg} is the speed for the best angle when gliding. In our work, because V_x depends only on the drag characteristics of the airframe and the propeller (and, in fact, only on the ratio between induced and parasite components), it plays the natural base role.

VIII. Conclusions

Three specific issues pertaining to this theory require further thought or experiment:

1) Π is defined under conditions of maximum power leaning while cruise performance usually assumes a recommended lean mixture. Our sample Cessna 172P POH (Ref. 7, pp. 4–18), quantifies recommended lean as 50°F rich of peak EGT. Because the engine manual (Ref. 7, pp. 3–13) shows that setting providing slightly over 99% of the optimum power, there is little to no problem in our case. In some other cases, a modest adjustment might be called for.

2) Does C, the bootstrap data plate altitude dropoff parameter, have the same value for the virtual derated engine as for the actual one? For practical purposes, it most likely does. Because that factor varies little among actual engines of widely varying displacement and power, the two are likely very close. In addition, no performance item depends critically on the value of C.

3) A more important question is, to what extent does Π correspond to the spatial position of the throttle control plunger? In other words, if one pulls back the throttle control 2 in. at 10,000 ft, thereby throttling back to, say $\Pi = 0.55$, then leaves the throttle control alone, will it still be $\Pi = 0.55$ at 2000 ft? Because of temperature expansion in the throttle and carburetor linkages, and the vagaries of volumetric efficiency, we doubt that Π will be precisely maintained constant. In a sample problem we assumed a constancy of Π, but we do not know whether that assumption is correct. A few simple experiments involving partial-throttle ceilings (because in those cases airspeed dependency is so easy to deal with) would settle the issue for at least a single airplane.

Supportive experiments are called for generally. There are simplifying bootstrap assumptions concerning torque and throttle, small flight-path angles, and the quadratic drag polar, plus a new set of assumptions connected to this partial-throttle extension. Any lengthy sorites of this type, no matter how well or convincingly argued, always requires experimental verification. Experiments, when all is said and done, are the final arbiter of the value of a theory. Small emendations to this theory may be needed in some places.

On the whole, however, the bootstrap approach to fixed-pitch propeller airplane performance is satisfyingly simple, and, where tested, has come up with substantially correct performance figures. Even in untested cases, it has come up with very reasonable ones. The theory is much better than any ersatz concoction of curve fits. Neither does its efficacy depend primarily on, e.g., the simple quadratic form of thrust as a function of airspeed. Sound aeronautical physics and engineering, starting with the linearized propeller polar itself, lies behind each bootstrap performance formula. With this latest extension to partial-throttle operations, alongside earlier treatments of gliding and full throttle flight in both wings-level and banked variants, the bootstrap approach has matured into a full and practicable theory. Using only nine parameters, fuel-consumption data, an easily obtained propeller chart, and five or fewer operating variables, the bootstrap approach gives the pilot good quantitative estimates of his or her airplane's steady-state performance.

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